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Intermittency in random optical layers at total reflection

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Abstract. We show that the reflection coefficient r of a succession of N random optical layers has a well defined, but sample-dependent, limit when $N \to \infty$. The probability distribution of r at total reflection is found to depend only on $(\mathrm{Im}\, r)/\xi^{2/3}$ where ξ^2 is the noise variance. The analytical expressions appearing in this localisation problem are found to be the same as the universal functions describing intermittency with noise.

1. Introduction

We wish to discuss the following physical problem: consider a succession of N layers such that the optical index of the ith layer is of the form $n_i = \bar{n}(1+\xi_i)$ where ξ_i is a (small) random variable such that $\langle \xi_i \rangle = 0$.

A light beam, sent from a medium with an optical index n_0 larger than $\bar{n}$, falls on the 'one-dimensional random medium' obtained by piling together N layers of thickness l (figure 1).

An interesting situation occurs when the incidence angle reaches the 'average critical angle' defined by

$$n_0 \sin \theta_c = \bar{n}.$$

Schematically, when ξ_i is positive the light beam is transmitted, while it is reflected when ξ_i is negative. What happens 'on average'? What can be said about the transmission and reflection coefficients as functions of the noise variance $\langle \xi_i^2 \rangle = \sigma$?

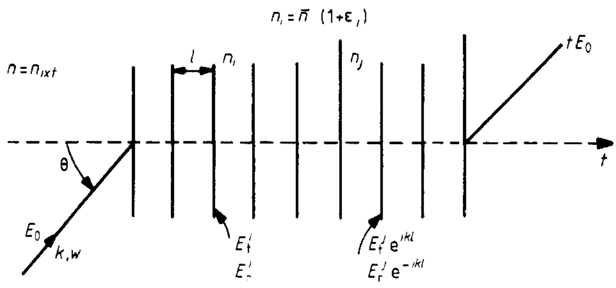

Figure 1. The system considered and the definition of the notation introduced in the text.

A problem of this kind has been considered for acoustic waves in a random-layered medium (Baluni and Willemsen 1984), but for a normal incidence, and only the frequency dependence of the transmission coefficient has been determined. The chain of random impedance studied by Akkermans and Maynard (1984) is formally equivalent to our problem but the 'critical incidence' (corresponding to resonance, $LC\omega^2 = 1$, but without dissipation—$R = 0$) has not been investigated. The problem we are interested in can also be cast in the form introduced by Derrida and Gardner (1984) for the one-dimensional Anderson problem. For this problem, the critical incidence corresponds to band-edge localisation. Here, we recover some of the results obtained by Derrida and Gardner (1984) and Halperin (1965) using another approach: we renormalise directly the random matrix product appearing in the problem.

The new results contained in this paper can be summarised in three points.

(i) The problem is presented in optical terms; the results given in this paper should be experimentally observed (Bouchaud and Daoud 1985).

(ii) We discuss the behaviour of *both* the transmission and reflection coefficients (only the transmission coefficient is usually considered, even though the reflection problem has been solved by Sulem (1973) for normal incidence). The reflection coefficient is shown to be sample dependent. The scaling with respect to the noise is obtained through a close analysis of Furstenberg's theorem.

(iii) We show that this localisation problem (expressed as a 2×2 random matrix product) is formally equivalent to the Pomeau–Manneville intermittency with *noise*. The matrix involved is in fact the one which appeared in the study of intermittent dynamics in billiards, introduced by the authors (Bouchaud and Le Doussal 1985).

This paper is divided into three parts. In § 2, we introduce the formalism with which we describe the optical problem and recall some results on random matrix products. In § 3, we apply those results to the transmission and reflection coefficients of the system. We show that the dependence on the mean square σ is non-analytic ($\sigma^{1/3}$). We provide other examples where the method is useful. In § 4, we explain the analogy between localisation and intermittency with noise. This analogy is used to describe the case θ_0 close to θ_c.

2. Formalism. Random matrix products

2.1. A physical discussion

As is well known, the transmission coefficient as a function of the number of layers N changes, in the pure case ($\xi_i \equiv 0$), from an oscillatory behaviour to an exponential decrease when θ_0 crosses the value θ_c. For $\theta_0 = \theta_c$, the decay is algebraic, which means that for $\theta_0 \to \theta_c^+$ the penetration depth diverges. This is nothing but the divergence of the effective wavelength in the medium. An interesting aspect of this transition is the divergence of the partial wave amplitude for $\theta_0 = \theta_c$—the finiteness of the physical fields is entirely due to interference and compensation between two infinite amplitudes.

A first simple approach to the disordered problem could be the following: consider the series of successive separating planes as independent 'mirrors', each one being characterised by a transmission coefficient t_n calculated as if only two semi-infinite media were present. In this approximation scheme, the transmission coefficient of the whole system is the product of the different t_n involved: multiple scatterings and interferences of light 'trapped' in a layer have been ignored. This leads to the following behaviour:

$$t \sim \exp(-\lambda(\sigma)N) \qquad \text{for large } N$$

with

$$\lambda(\sigma) \sim \sigma \qquad \text{if } \theta_0 < \theta_c,$$
$$\lambda(\sigma) \sim \sigma^{1/2} \qquad \text{for } \theta_0 = \theta_c.$$

This 'mean field' calculation leads to the correct scaling for $\theta_0 < \theta_c$ but is not successful in the case $\theta_0 = \theta_c$, for which $\lambda(\sigma)$ behaves rather like $\sigma^{1/3}$ as will be shown in § 3. As usual in phase transition problems, the reason for this breakdown is simple: as the mean effective wavelength diverges for $\theta_0 = \theta_c$, it is not justified to consider that the successive planes are far apart. Let us finally mention that this 'mean field' approach is quite the analogue of the random phase model introduced by Anderson *et al* (1980).

2.2. Transfer matrices

We shall denote by E_i^t (resp. E_i^r) the electric field propagating in the $z > 0$ (resp. $z < 0$) direction in the ith layer. The polarisation of the incident plane wave is such that the electric field is perpendicular to the incidence plane (see figure 1).

Fresnel's equations allow us to write recurrence equations in the variables

$$X_i = E_i^t + E_i^r, \qquad Y_i = j(E_i^t - E_i^r)k_i l,$$

with

$$j^2 = -1,$$
$$k_i^2 = \bar{k}^2[(1-(n_0/\bar{n})^2 \sin^2\theta_0) + 2\xi_i],$$
$$\bar{k} = \bar{n}\omega/c.$$

The recurrence relation reads:

$$\begin{pmatrix} X_{i+1} \\ Y_{i+1} \end{pmatrix} = Q_i \begin{pmatrix} X_i \\ Y_i \end{pmatrix}$$

with

$$Q_i = \begin{bmatrix} \cos k_i l & (lk_i)^{-1} \sin k_i l \\ -lk_i \sin k_i l & \cos k_i l \end{bmatrix} \qquad (i \geq 2),$$

$$Q_1 = \mathbb{1}.$$

We shall consider from now on the case $\theta_0 = \theta_c$. (The case θ_0 close to θ_c will be discussed in § 4.2.) Up to the first order in ξ, this leads to

$$Q_i = \begin{bmatrix} 1 + \frac{1}{2}\xi_i & 1 \\ \xi_i & 1 + \frac{1}{2}\xi_i \end{bmatrix} \tag{2.1}$$

with a new $\xi_i = -2(\bar{k}l)^2 \xi_i$.

The equations defining the (complex) transmission and reflection coefficients (t, r) thus read

$$\begin{pmatrix} t \\ \mathrm{i}k_0 t \end{pmatrix} = \left(\prod_{i=1}^{N+1} Q_i\right) \begin{pmatrix} 1 + r \\ \mathrm{i}k_0(1-r) \end{pmatrix}. \tag{2.2}$$

Every matrix Q_i (and thus their product) satisfies

$$\det Q_i = 1. \tag{2.3}$$

This property ensures that the electromagnetic energy is conserved, and it is straightforward to prove that, if Q_i is real†, (2.3) is equivalent to

$$|r|^2 + |t|^2 = 1. \tag{2.4}$$

For $\xi_i = 0$, Q_i takes the form $\left[\begin{smallmatrix}1 & 1\\ 0 & 1\end{smallmatrix}\right]$ which is exactly the linear part of a two-dimensional intermittent mapping that we introduced to study the motion of a point particle in billiards between two perfectly reflecting circles close to contact (Bouchaud and Le Doussal 1985).

2.3. Product of random matrices. Furstenberg's theorem

Let $T_N = \Pi_{i=1}^{N+1} Q_i$. Furstenberg's theorem (Furstenberg 1963) states that the angle between $T_N\boldsymbol{u}$ and $T_N\boldsymbol{u}'$ (where $\boldsymbol{u}$ and $\boldsymbol{u}'$ are two arbitrary vectors) goes to zero exponentially with N:

$$\lim_{N\to\infty} \frac{1}{N} \log\langle T_N\boldsymbol{u}, T_N\boldsymbol{u}'\rangle = -2\lambda$$

($\langle \ldots \rangle$ denotes a scalar product). λ is called the Lyapounov exponent of the matrix product. By applying this theorem to T_N and T_N^{-1} (which gives the same λ by virtue of equation (2.3)) and taking the shape of the Q_i into account, it is plausible (and experimentally confirmed) that *every* coefficient of T_N diverges according to the same law:

$$a_N \sim b_N \sim c_N \sim d_N \sim e^{\lambda N}, \tag{2.5}$$

$$T_N = \begin{bmatrix} a_N & b_N \\ c_N & d_N \end{bmatrix} \qquad (a_N, b_N, c_N, d_N) \in \mathbb{R}^4. \tag{2.6}$$

T_N can be cast in diagonal form in $\mathbb{R}$: it can be written

$$\begin{bmatrix} \alpha_N\, e^{\lambda N} & 0 \\ 0 & \alpha_N{}^{-1}\, e^{-\lambda N} \end{bmatrix}$$

(with $\alpha_N \simeq O(1)$) defining, as in dynamical systems, a stable and an unstable manifold. The direction of the stable manifold of T_N defines, in the plane, an angle θ_N, which satisfies

$$\lim_{N\to\infty} \frac{\theta_N}{N} = \omega, \tag{2.7}$$

called the rotation number of the product (see, for example, Ruelle 1985). From (2.2) and (2.6), we get

$$t = 2[(a_N + d_N) + i(c_N/k_0 l - k_0 l b_N)]^{-1} \sim e^{-\lambda N}. \tag{2.8}$$

This exponential decrease of the transmission coefficient is nothing but another aspect of Anderson's localisation (in one dimension).

If we call $\varphi = 1 - r$, we also obtain, from (2.2):

$$\varphi = 2\frac{a_N + i c_N/k_0 l}{a_N + d_N + i(c_N/k_0 l - k_0 l b_N)}. \tag{2.9}$$

† This is not the case when absorption is taken into account.

Condition (2.5) thus implies that φ reaches a limit φ^*—*a priori* different from zero—when $N \to \infty$.

We shall see in the following that this limit is extremely well defined experimentally (which confirms hypothesis (2.5)), but is sample dependent; that is, φ^* is dependent on the particular realisation of the ξ_i (φ^* is not self-averaging).

Note that φ^* is zero when $\xi_i \equiv 0$, and that condition (2.4) imposes that

$$\mathrm{Re}\,\varphi^* = \tfrac{1}{2}(\mathrm{Im}\,\varphi^*)^2 \qquad \text{when } \varphi^* \ll 1.$$

The structure of T_N can be further discussed: due to time reversal invariance, T_N can always be written, in the

$$\begin{pmatrix} E_i^{\mathrm{t}} \\ E_i^{\mathrm{r}} \end{pmatrix}$$

basis, in the form (Pichard 1984)

$$T_N = \begin{bmatrix} \alpha & \beta \\ \beta^* & \alpha^* \end{bmatrix}$$

which allows us to identify $\alpha = 1/t^*$, $\beta = -r^*/t^*$.

In the $\binom{X_i}{Y_i}$ basis, this implies that T_N is equal to

$$\frac{1}{2|t|^2}\begin{bmatrix} t+t^*-(r^*t+rt^*) & (\mathrm{i}/k_0l)(t-t^*)+rt^*-r^*t \\ \mathrm{i}k_0l(t-t^*)+(rt^*-r^*t) & (t+t^*)+(r^*t+rt^*) \end{bmatrix}. \tag{2.10}$$

We now wish to show in a different way that

$$T_N = \mathrm{e}^{\lambda N} P_N \tag{2.11}$$

where P_N is exponentially close to a projector $P_N^2 \simeq P_N$.

The most general form of a 2×2 projector is

$$P_{\gamma\delta} = \begin{bmatrix} \frac{1}{2}(1-\gamma) & (1-\gamma^2)/2\delta \\ \frac{1}{2}\delta & \frac{1}{2}(1-\gamma) \end{bmatrix}. \tag{2.12}$$

We are thus led to the following identifications:

$$\mathrm{e}^{\lambda N} \sim (t+t^*)/|t|^2 = g$$

which we factorise in (2.10),

$$\gamma \simeq -\frac{r^*t+rt^*}{t+t^*}, \qquad \delta = \mathrm{i}(k_0l)\frac{t-t^*+rt^*-r^*t}{t+t^*},$$

so that

$$g\frac{1-\gamma^2}{2\delta} = \frac{1}{2\mathrm{i}k_0l|t|^2}\,\frac{(t+t^*)^2-(rt^*+r^*t)^2}{t-t^*+rt^*-r^*t}.$$

In order to recover the corresponding term of T_N, one needs that $|r|^2 = 1$, thus $|t|^2 = 0$. This proves (2.11) since $|t|^2 \simeq \mathrm{e}^{-2\lambda N}$.

2.4. *Conjecture on T_N extrapolating between λN small and large λN*

From the last remark and the form of Q_i when $\xi_i \equiv 0$ we propose a conjecture concerning the structure of T_N very similar to the conjecture of Anderson *et al* (1980) on the resistivity of a one-dimensional disordered array: $\rho = e^{\alpha L} - 1$, where L is the 'length' of the sample and α^{-1} is the localisation length. This expression leads to $\rho \simeq \alpha L$ when $\alpha L \ll 1$ and $\rho \simeq e^{\alpha L}$ when $\alpha L \gg 1$.

Let us write $T_N(\xi) = T(N, \xi)$ in the form (throughout we will denote $\xi = \sqrt{\sigma}$)

$$T_N(\xi) = \mu(N, \xi)(P'(\xi) + N^{-1}G(\xi) + O(N^2))$$

where $\mu(N, \xi)$ contains the 'large' N dependence and G, $P' = O(1)$ with $P'(0) = \left[\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right]$ (nilpotent matrix), $G(0) = \mathbb{1}$. Neglecting terms in ξ/N, the identification between $T(2N, \xi)$ and $T(N, \xi)^2$ leads to

$$P'(\xi)^2 = \lambda_N(\xi)P'(\xi)$$

with

$$\lambda_N(0) = 0, \qquad \lambda_N = \mu(2N, \xi)/\mu(N, \xi)^2 - 2/N.$$

The natural conjecture on $T(N, \xi)$ thus reads:

$$T(N, \xi) = [\mathbb{1} + P'(\xi)]^N \qquad \text{with } P'(\xi)^2 = \lambda(\xi)P'(\xi), \tag{2.13}$$

where $\lambda(\xi)$ is N dependent, which allows us to recover (Anderson *et al* 1980):

$$\text{for } \lambda(\xi)N \ll 1, \qquad T_N \simeq \mathbb{1} + NP'(\xi), \tag{2.14a}$$

$$\text{for } \lambda(\xi)N \gg 1, \qquad T_N \simeq (P'(\xi)/\lambda(\xi))\, e^{N\lambda(\xi)}. \tag{2.14b}$$

$\lambda(\xi)$ is thus the Lyapounov exponent and $P(\xi) = P'(\xi)/\lambda(\xi)$ is a projector ($P^2 = P$) due to property (2.13). We thus recover (2.11). Let us nevertheless stress once more that $P(\xi)$ is still N dependent. $P(\xi)$ can be parametrised by (γ_N, δ_N) according to (2.12). The scaling behaviour of γ, δ will be studied in § 3. In fact, (2.13) describes the limits $\lambda N \ll 1$ and $\lambda N \gg 1$ correctly, but must be taken with care if one wants to describe the crossover. The validity of (2.13) in this region should be compared with the random phase model of Anderson *et al* (1980), which has been recently discussed by Pichard (1985).

3. Renormalisation. Scaling for $\lambda(\xi)$ and $\varphi^*(\xi)$

3.1. Scaling laws

As in a previous paper (Bouchaud and Le Doussal 1985), where we renormalised the dynamics of a point particle between two circles, we introduce here the matrix $A = \left[\begin{smallmatrix} 1 & 0 \\ 0 & 1/2 \end{smallmatrix}\right]$ which satisfies the renormalisation condition

$$A^{-1}Q^2(\xi = 0)A = Q(\xi = 0). \tag{3.1}$$

If ξ_i and ξ_{i+1} are two (uncorrelated) Gaussian noises with variance σ, their sum is Gaussian with variance 2σ. We thus obtain

$$A^{-1}Q_i(\xi)Q_{i-1}(\xi)A \simeq Q_i(2\sqrt{2}\xi). \tag{3.2}$$

$T(N, \xi)$ can be transformed into

$$T(N, \xi) = AT(\tfrac{1}{2}N, 2\sqrt{2}\xi)A^{-1}. \tag{3.3}$$

Taking the trace of this last equation, and using (2.14b), we obtain, for large N,

$$\lambda(\xi) = \tfrac{1}{2}\lambda(2\sqrt{2}\xi) \qquad \text{or} \qquad \lambda(\xi) \sim \xi^{2/3}. \tag{3.4}$$

Figure 2 illustrates the physical meaning of equation (3.3): as usual in Kadanoff's renormalisation, the layers are paired together and the equivalent index of those pairs is determined. So if r' and t' are the coefficients relative to a system of N layers with index $(\bar{n}+2\sqrt{2}\xi_i)$ and r, t those of the 'decimated' system of $N/2$ layers with index $(\bar{n}+\xi_i)$, one has the renormalisation equations:

$$A^{-1}\begin{pmatrix} t' \\ \mathrm{i}k_0 l t' \end{pmatrix} = T\left(\frac{N}{2}, \xi\right) A^{-1}\begin{pmatrix} 1+r' \\ \mathrm{i}k_0 l(1-r') \end{pmatrix},$$

$$\begin{pmatrix} t \\ \mathrm{i}k_0 l t \end{pmatrix} = T\left(\frac{N}{2}, \xi\right)\begin{pmatrix} 1+r \\ \mathrm{i}k_0 l(1-r) \end{pmatrix}. \tag{3.5}$$

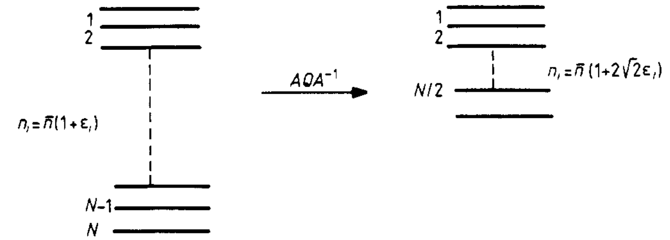

Figure 2. The geometrical meaning of the renormalisation transformation $T(N,\xi)\to T(\frac{1}{2}N, 2\sqrt{2}\xi)$.

From (3.4) we obtain N^{-1} Log $t \sim \xi^{2/3}$, which is Derrida and Gardner's (1984) result, and is to be compared with the 'mean field' prediction of § 2.1. Equations (3.5) together with the condition det $T(\frac{1}{2}N, \xi)=1$ allow us to obtain

$$\varphi(N, \xi) = \tfrac{1}{2}\varphi(\tfrac{1}{2}N, 2\sqrt{2}\xi).$$

Thus φ^* satisfies the same relation as λ. With the equation Re $\varphi^* = \frac{1}{2}(\mathrm{Im}\ \varphi^*)^2$ we finally get

$$\varphi^* \simeq \mathrm{i}\alpha\xi^{2/3} + \tfrac{1}{2}\alpha^2\xi^{4/3} \tag{3.6}$$

where α is a (sample-dependent) real number.

A more straightforward approach, making use of the conjecture developed in § 2.4, can be given. The consistency condition

$$\lim_{\xi\to 0} \lambda(\xi)P(\xi) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},$$

together with $\lambda(\xi) \sim \xi^{2/3}$, imposes that

$$P(\xi) = \begin{bmatrix} \frac{1}{2}(1-\gamma) & \xi^{-2/3}(1-\gamma)/2\delta \\ \xi^{2/3}\delta/2 & \frac{1}{2}(1+\gamma) \end{bmatrix}$$

where δ, γ are bounded and sample dependent (and N dependent). It is then easy, using the explicit expression (2.9) for φ, to derive (3.6).

We finally formulate two remarks.

(1) If the noise is not Gaussian but of a ν-Levy type (ν is the largest number such that $\langle\xi^\nu\rangle < +\infty$, the exponent $\frac{2}{3}$ has to be transformed into $\nu/(\nu+1)$ ($\nu=2$ corresponds to the Gaussian case).

(2) The rotation number defined by (2.7) is easily seen to satisfy the same scaling equation as $\lambda(\xi)$. One thus has $\omega\ (\xi) \sim \xi^{2/3}$.

This rotation number is linked with the 'imaginary part' of the Lyapounov exponent of Derrida and Gardner (1984), which in turn gives the energy density of states in Anderson's model (see Thouless 1972).

3.2. *Physical interpretation.*

The transmission coefficient goes to zero exponentially: as we already said, this is the well known result of 1D localisation. The localisation length is non-analytic in the noise: this is the consequence of the competition between an oscillatory and an exponentially decaying behaviour at criticality. The reflection coefficient r is not equal to 1: for a given sample $\varphi = 1 - r$ converges towards a well defined limit when $N \to \infty$ (cf figure 3). But for a different realisation of the ξ_i (for a given σ), these limits are not the same (cf figure 3). Formula (3.6) must then be interpreted in the following way: one has to introduce a probability distribution p over the samples. It depends on φ^* and σ only through $(\mathrm{Im}\ \varphi^*)/\sigma^{1/3}$:

$$p(\varphi^*, \sigma) = \phi((\mathrm{Im}\ \varphi^*)/\sigma^{1/3}). \tag{3.7}$$

The fact that φ^* is not self-averaging (sample dependent) is easy to understand. As the amplitude of the beam decreases exponentially, the far-away layers are only weakly probed, and in fact only the actual realisation of ξ_i on the first few layers is important. This can be enlightened by the following 'toy model': let x_i be a random variable of variance σ. If one studies the following weighted sum, representing the 'superposition' of the contributions of each layer: $x = \Sigma_k\, x_k\, \mathrm{e}^{-\lambda k}$, it is easy to see that the distribution $p(x)$ is also a Gaussian (and not a δ) of variance $\sigma/2\lambda$ (for small λ). Since here $\lambda \sim \sigma^{1/3}$, this model provides a Gaussian probability distribution for Im φ^*, of the

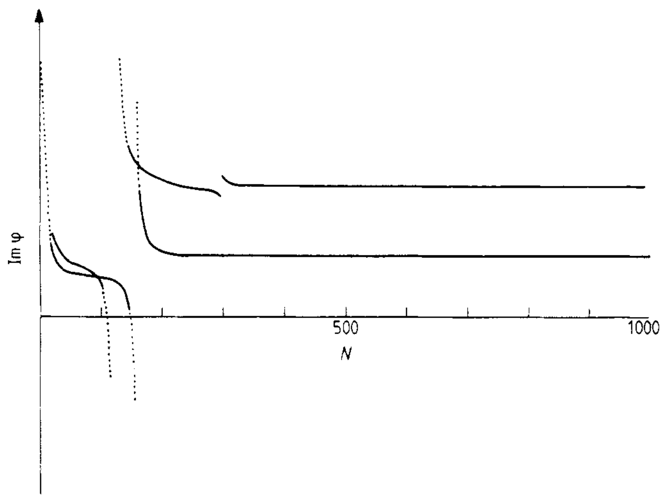

Figure 3. We show in this figure $\varphi = 1 - r$ as a function of the number of layers N. The two curves correspond to two different realisations of ξ_i with the *same* $\langle \xi^2 \rangle$, showing the fact that φ reaches a well defined but sample-dependent limit. Note also (see § 4.2) the 'jumps' corresponding to chaotic bursts of the underlying intermittent mapping.

form (3.7). In fact ϕ is not exactly Gaussian and can be determined exactly (Sulem 1973). It is given in § 4 and drawn in figure 5.

For normal incidence, the same theory can be developed (matrix renormalisation). The exponent $\frac{1}{3}$ must be replaced by 1 in the above formulae, and follows the prediction of the 'mean field' theory given in § 2.1.

Let us finally mention that φ^* should be rather easily reached through interference between incident and reflected waves.

3.3. *Other physical examples for which the method is useful*

A certain number of physical problems can be described as mappings or differential equations involving the matrix $Q_0 = \left[\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right]$. This matrix is interesting in two respects.

(i) As

$$\begin{bmatrix} 1 & 1 \\ x & 1 \end{bmatrix} = \exp\left(x\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}\right) \tag{3.8}$$

the descriptions in terms of differential equations and iterated maps are equivalent (divergence of a 'length' scale). $x = 0$ corresponds to an intermittency *threshold* (cf Bouchaud and Le Doussal 1985 and § 4.2); and equation (3.8) means that the process near this threshold is *infinitely divisible* (this is not the case, for instance, near Feigenbaum's fixed point).

(ii) The product of matrices of type (3.8)—with x possibly random—leads to scaling laws characterised by exponents which can be calculated using property (3.1.).

We thus give a few physical examples for which scaling laws obtained through (3.1) are identical.

(a) Scaling laws (as functions of geometrical parameters) on Lyapounov exponents in different types of billiards (diamond (Bouchaud and Le Doussal 1985, Benettin 1984); stadion (Benettin 1984)).

(b) Calculation of the Lyapounov exponent λ (Kolmogorov entropy) associated with the motion of a particle in an inhomogeneous magnetic field (Rechester *et al* 1979). $\lambda \sim b^{2/3}$ where b is a measure of the field's inhomogeneity. (In this case, the perturbation of Q_0 is a different, but the 'renormalisation' transformation is the same as in case (a).)

(c) NMR spectrum of a particle with spin diffusing on a line according to a Brownian motion in a linearly varying magnetic field. The problem can be written in the following way:

$$x_{i+1} = x_i + D^{1/2}\xi_i, \qquad \theta_{i+1} = \theta + gbx_i, \tag{3.9}$$

where x_i is the position, ξ_i a Gaussian process, D the diffusion coefficient, θ_i is the angle of a transverse component of the spin and b the field gradient.

The frequency distribution (NMR spectrum) is $p(\omega)$ with

$$\omega = \lim_{N\to\infty} \frac{\theta_N}{N}.$$

The above formalism allows us to obtain the width of $p(\omega)$ as

$$\langle\omega^2\rangle^{1/2} \sim D^{1/3}(gb)^{2/3}$$

which is the classical result (Abragam 1960).

4. Link between localisation and intermittency

4.1. Intermittency: main features

Pomeau–Manneville's intermittency (Pomeau and Manneville 1980a, b) schematically models the dynamics of a variable whose behaviour is characterised by a succession of 'laminar', quasi-periodic eras and chaotic bursts, as encountered in the transition to turbulence. The mapping describing the dynamics of some Poincaré section of the flow has, in the laminar region, the following generic form:

$$x_{i+1} = f_\mu(x_i) = \mu + x_i - x_i^2 + \dots, \qquad \mu \text{ small}$$

(see figure 4). When x_i leaves the 'channel', a chaotic burst appears until non-linear terms reinject the 'particle' in the channel. This mapping has been fully discussed (Pomeau and Manneville 1980a, b, Hirsh *et al* 1982) in the deterministic case (μ = constant) as well as in the probabilistic case (Eckmann *et al* 1981),

$$f_{\mu,\xi} = \mu + \xi_i + x_i - x_i^2 + \dots, \tag{4.1}$$

where ξ_i is a random 'noise'.

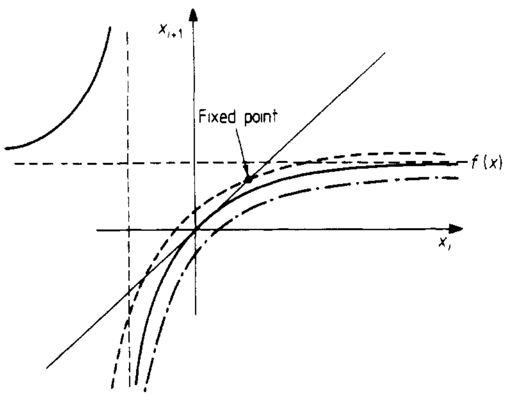

Figure 4. An archetype of intermittent mapping. Bold curves, the function $f_\mu = 2 + \mu + 1/x$ at threshold ($\mu = 0$). Dotted curve, the effect of a positive (or negative) noise ξ and the appearance of a transient fixed point (for $\xi > 0$).

Two interesting quantities can be defined in this problem.

(i) The laminar time T_L, i.e. the number of successive iterations the particle spends in the channel.

(ii) The Lyapounov exponent λ, defined for any mapping as

$$\lambda = \lim_{N\to\infty} N^{-1} \log|\partial f^N/\partial x|. \tag{4.2}$$

Assuming an ergodic property†, those two quantities can be expressed through the asymptotic measure of the problem, $p_\infty^{\mu,\xi}$, as

$$T_L^{-1} \sim \int_{-\infty}^{-1} p_\infty^{\mu,\xi}(x)\, dx, \tag{4.3a}$$

† A property which may not be true of $\xi_i \equiv 0$ (see discussion in § 4.3).

$$\lambda = \int_{-\infty}^{+\infty} p_\infty^{\mu,\xi}(x) \log|f'(x)|\,\mathrm{d}x \simeq \int x p_\infty^{\mu,\xi}(x)\,\mathrm{d}x. \tag{4.3b}$$

Eckmann *et al* (1981) showed that, for small ξ and μ,

$$p_\infty^{\mu,\xi}(x) = C\xi^{-2/3} \exp\left[-2\left(\frac{t^3}{3} - \frac{\mu}{\xi^{4/3}}t\right)\right] \int_{-\infty}^{t} \mathrm{d}t' \exp\left[2\left(\tfrac{1}{3}t'^3 - \frac{\mu}{\xi^{4/3}}t'\right)\right] \tag{4.4}$$

with $x = \xi^{2/3}t$, which gives, for $\xi = 0$, the well known result

$$T_{\mathrm{L}} \sim \mu^{-1/2} \qquad (\mu > 0)$$

and, for $\mu = 0$, $T_{\mathrm{L}} \sim \xi^{-2/3}$.

T_{L} is thus expressed as $T_{\mathrm{L}}(\xi, \mu) = \mu^{-1/2} F(\xi^{4/3}/\mu)$ where F is a *universal* function for one-dimensional intermittency with noise (Eckmann *et al* 1981).

4.2. Localisation

In this section we consider the one-dimensional Schrödinger equation in a random potential

$$\psi_{i+1} - 2\psi_i + \psi_{i-1} = (E - V_i)\psi_i. \tag{4.5}$$

As explicitly shown in the appendix, this is equivalent to studying the product of random matrices (2.1). Introducing $R_{i+1} = \psi_{i+1}/\psi_i$, we have

$$R_{i+1} = 2 + \mu + \xi_i - 1/R_i \qquad \text{with} \qquad \mu \equiv -E,\ \xi_i \equiv V_i.\dagger \tag{4.6}$$

For $\xi_i = 0$, the function $f(x) = 2 - x^{-1}$ has exactly the shape of an intermittent Pomeau-Manneville function (figure 4) including the reinjection process (see § 4.3). When R_i is close to 1 ($R_i = 1 + x_i$, $x \ll 1$), one has

$$x_{i+1} = \xi_i + x_i - x_i^2 + \mu + \dots \tag{4.7}$$

which must be compared with (4.1).

The localisation length of the problem is defined through

$$\lambda = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \log|R_i|. \tag{4.8}$$

Introducing the distribution $p(R)$ of the variable R_i verifying a Dyson (1953) relation,

$$p(R) = \int \mathrm{d}\xi\, p(\xi) \int \mathrm{d}R'\, \delta(R - 2 - \xi - \mu + 1/R')p(R') \tag{4.9}$$

($p(\xi)$ is the noise distribution), we obtain the analogue, for the mapping (4.6), of the asymptotic measure $p_\infty^{\mu,\xi}$ defined in § 4.1. This probability distribution $p(R)$ allows us to compute the inverse localisation length (Derrida and Gardner 1984) as

$$\lambda = \frac{\int p(R) \log R\, \mathrm{d}R}{\int p(R)\,\mathrm{d}R} = \tfrac{1}{2}\xi^{2/3} \frac{\int_0^\infty \mathrm{d}t\, t^{1/2} \exp(-\frac{1}{6}t^3 + 2\mu t/\xi^{4/3})}{\int_0^\infty \mathrm{d}t\, t^{-1/2} \exp(-\frac{1}{6}t^3 + 2\mu t/\xi^{4/3})}. \tag{4.10}$$

In the continuous limit one can show (Sulem 1973) that for small φ^*,

$$\operatorname{Im} \varphi^* \simeq 2\dot{\psi}/\psi (= 2\varkappa_N).$$

† In the optical problem $\mu \sim \theta_c - \theta$.

This is due to the fact that for a complex initial R_0, R_N obtained through (4.6) is real when $N \to \infty$. We can thus further specify the distribution law of Im φ^*: it is simply given by the distribution of x: $p_\infty^{\mu,\xi}(x)$ written in (4.4). This is consistent with (3.7) obtained through scaling, as $p_\infty^{\mu,\xi}(x)$ only depends on $t = x/\xi^{2/3}$. In figure 5 we give a sketch of this function for $\mu = 0$.

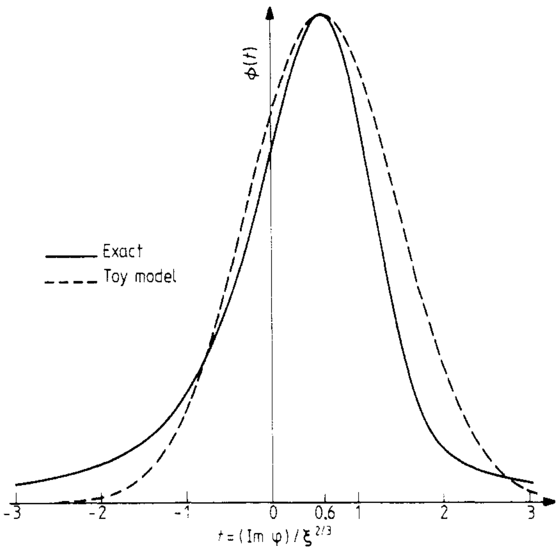

Figure 5. The distribution law of $p(\varphi^*)$ as a function of the scaled variable (Im $\varphi^*)/\xi^{2/3}$.

Thus, we point out that, in order to study a localisation problem, one is naturally led to introduce an *intermittent* mapping of the form (4.7). There is a close correspondence between the interesting quantities relevant in dynamical system analysis and the physically important observables in localisation. Indeed, the inverse localisation length defined by (4.8) is *exactly* minus one half the Lyapounov exponent λ' of the intermittent mapping (4.6) given by (4.3*b*) (see § 4.3 below); in the same way, the imaginary part of λ†, defined in Derrida and Gardner (1984), counts the number of sign changes of R_i—that is, the number of chaotic bursts in the intermittent problem. As this number is simply the inverse of the 'laminar time' usually considered, we have

$$1/T_{\mathrm{L}} \sim \mathrm{Im}\ \lambda.$$

Thus, according to Derrida and Gardner (1984), we have

$$T_{\mathrm{L}} \sim \xi^{-2/3} \int_0^\infty t^{-1/2}\,\mathrm{d}t \exp\left(-\frac{t^3}{6} + \frac{2\mu t}{\xi^{4/3}}\right) \tag{4.11}$$

which is nothing but the formula (4.1) appearing in Eckmann *et al* (1981) for the laminar time of a general one-dimensional intermittent mapping with noise.

This analogy allows us in particular:

(1) to interpret the 'jumps' in the curve Im $\varphi(N)$ (figure 3) as *chaotic bursts*. These bursts also correspond to moments where the coefficients c_N of the matrix T_N changes sign;

† This imaginary part is also the rotation number of the flow (2.7).

(2) to predict the scaling behaviours for θ near the critical angle θ_c by writing everywhere

$$\mu \sim \theta_c - \theta.$$

4.3. The limit $\xi \to 0$

The following point needs clarification: formula (4.10) leads, in the limit $\xi \to 0$, to:

(i) $\lambda' = 0$ for $\mu < 0$ (*no* fixed point for the mapping (4.6));

(ii) $\lambda' = -\sqrt{\mu}$ for $\mu > 0$ (one stable fixed point $x_0 \sim \sqrt{\mu}$).

Nevertheless, we are accustomed to think that the case $\mu < 0$ (no fixed point) is the 'chaotic phase'—its stochasticity being characterised by a positive Lyapounov exponent growing as $\sqrt{-\mu}$. A closer analysis shows that this positive exponent is entirely due to the chaotic reinjection processes (bursts) giving a contribution to λ' of order $\lambda' \sim$ number of chaotic bursts $\sim 1/T_L \sim \sqrt{-\mu}$ (following (4.11)). The channel contribution to λ is *strictly* zero. The reinjection process contained in the $1/R$ part of mapping (4.6) is not 'stochastic' enough; this is linked with the fact that (4.6) without noise is *not ergodic* (it is in fact periodic whenever $\sqrt{-\mu}$ is rational). Without noise, the asymptotic measure $p(R)$ loses its meaning.

On the contrary, when there exists a fixed point, λ' is negative as it should be (contraction of phase space). When the noise comes in, formula (4.10) also leads to a negative λ'; this is because *only* the existence (even temporary—when $\xi_i + \mu > 0$) of a fixed point contributes (negatively) to λ': the channel contribution is zero and the bursts contribution cannot be calculated with the local analysis (around $R = 1$) developed by Derrida and Gardner (1984) and Eckmann *et al* (1981). It may be that, for another mapping, a sufficiently chaotic reinjection and the transient fixed point compete in the determination of λ'.

5. Conclusion

In this paper, we showed that the reflection coefficient r of a set of N random layers has a well defined (but sample-dependent) limit when $N \to \infty$. This coefficient is not equal to one; instead, the distribution law of $\varphi = 1 - r$ depends only on $(\mathrm{Im}\ \varphi)/\xi^{2/3}$ (where ξ^2 is the noise variance) if the incidence angle is the mean total reflection angle. This scaling is quite easily obtained with the help of a close analysis of the expected asymptotic form of the random matrix product (§ 2.4), and the distribution of Im φ can be specified entirely.

We also unveiled a strong link between intermittency and localisation, hoping that certain physical concepts important in localisation may prove useful in fluid mechanics. Let us note that in fact *intermittency* follows from a wave equation $\Delta\psi = -E\psi$ near $E = 0$, or, seen differently, when a squared mass vanishes.

Intermittency might thus be deeply connected to second-order phase transitions; the problem dealt with in this paper is in fact, in a way, such a phase transition, where the order parameter is the localisation length and the noise an external field.

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Appendix

The Schrödinger equation (4.5) can be written

$$\begin{pmatrix} \psi_{i+1}-\psi_i \\ \frac{1}{2}(\psi_{i+1}+\psi_i) \end{pmatrix} = \begin{bmatrix} 1+\frac{1}{2}\xi_i & \xi_i \\ 1+\frac{1}{4}\xi_i & 1+\frac{1}{2}\xi_i \end{bmatrix} \begin{pmatrix} \psi_i-\psi_{i-1} \\ \frac{1}{2}(\psi_i+\psi_{i-1}) \end{pmatrix}$$

with $\xi_i \equiv V_i - E$. This is very close to the form (2.1) for Q_i and the difference is clearly irrelevant for small ξ_i.

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